

# Explicit real-part estimates for high order derivatives of analytic functions

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**Abstract.** The representation for the sharp constant  $K_{n,p}$  in an estimate of the modulus of the  $n$ -th derivative of an analytic function in the upper half-plane  $\mathbb{C}_+$  is considered. It is assumed that the boundary value of the real part of the function on  $\partial\mathbb{C}_+$  belongs to  $L^p$ . The representation for  $K_{n,p}$  comprises an optimization problem by parameter inside of the integral. This problem is solved for  $p = 2(m+1)/(2m+1-n)$ ,  $n \leq 2m+1$ , and for some first derivatives of even order in the case  $p = \infty$ . The formula for  $K_{n, 2(m+1)/(2m+1-n)}$  contains, for instance, the known expressions for  $K_{2m+1,\infty}$  and  $K_{m,2}$  as particular cases. Also, a two-sided estimate for  $K_{2m,\infty}$  is derived, which leads to the asymptotic formula  $K_{2m,\infty} = 2((2m-1)!!)^2/\pi + O(((2m-1)!!)^2/(2m-1))$  as  $m \rightarrow \infty$ . The lower and upper bounds of  $K_{2m,\infty}$  are compared with its value for the cases  $m = 1, 2, 3, 4$ . As applications, some real-part theorems with explicit constants for high order derivatives of analytic functions in subdomains of complex plane are described.

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## 0 Introduction

In this paper we deal substantially with the coefficient  $K_{n,p}(\alpha)$  in the inequality

$$|\Re\{e^{i\alpha} f^{(n)}(z)\}| \leq \frac{K_{n,p}(\alpha)}{(\Im z)^{n+\frac{1}{p}}} \|\Re f\|_p, \quad (0.1)$$

where  $z$  is a point in the half-plane  $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ . Here  $f$  is an analytic function in  $\mathbb{C}_+$  represented by the Schwarz formula

$$f(z) = \frac{1}{\pi i} \int_{\infty}^{\infty} \frac{\Re f(\zeta)}{\zeta - z} d\zeta \quad (0.2)$$

and such that the boundary values on  $\partial\mathbb{C}_+$  of the real part of  $f$  belong to the space  $L^p(-\infty, \infty)$ ,  $1 \leq p < \infty$ .

Here and in what follows we adopt the notation  $\|\Re f\|_p$  for  $\|\Re f|_{\partial\mathbb{C}_+}\|_p$ , where  $\|\cdot\|_p$  stands for the norm in  $L^p(-\infty, \infty)$ . Note that the value  $K_{n,\infty}(\alpha)$  is obtained by passage to the limit of  $K_{n,p}(\alpha)$  as  $p \rightarrow \infty$ .

Inequality (0.1) with the best possible coefficient in front of  $\|\Re f\|_p$  was obtained by Kresin and Maz'ya [8]. In [8] it was shown that

$$K_{n,p}(\alpha) = \frac{n!}{\pi} \left\{ \int_{-\pi/2}^{\pi/2} \left| \cos \left( \alpha - (n+1)\varphi + \frac{n\pi}{2} \right) \right|^q \cos^{(n+1)q-2} \varphi d\varphi \right\}^{1/q}, \quad (0.3)$$

where  $1/p + 1/q = 1$ . So, the sharp constant  $K_{n,p}$  in the inequality

$$|f^{(n)}(z)| \leq \frac{K_{n,p}}{(\Im z)^{n+\frac{1}{p}}} \|\Re f\|_p \quad (0.4)$$

is given by

$$K_{n,p} = \max_{\alpha} K_{n,p}(\alpha). \quad (0.5)$$

Note that inequalities (0.1) and (0.4) for analytic functions belong to the class of sharp real-part theorems (see Kresin and Maz'ya [6] and references there) which go back to Hadamard's real-part theorem [2].

The present article extends the topic of papers by Kresin and Maz'ya [7, 8]. In [7] the explicit formulas for  $K_{0,p}$  for  $p \in [1, \infty)$  and for  $K_{1,p}$  for  $p \in [1, \infty]$  were found. In [8] the case of  $n \geq 2$  was considered and the explicit formulas for  $K_{n,p}$  were derived for  $n = 2m+1, 2, 4$  and  $p = \infty$  as well as for arbitrary  $n$  and  $p = 1, 2$ . Namely, in [8] it was shown that

$$K_{2m+1,\infty} = \frac{2}{\pi} \frac{((2m+1)!!)^2}{2m+1}, \quad m = 0, 1, 2, \dots, \quad (0.6)$$

$$K_{2,\infty} = \frac{3\sqrt{3}}{2\pi}, \quad K_{4,\infty} = \frac{3}{4\pi} (16 + 5\sqrt{5}), \quad (0.7)$$

and

$$K_{n,1} = \frac{n!}{\pi}, \quad K_{n,2} = \sqrt{\frac{(2n)!}{2^{2n+1}\pi}}. \quad (0.8)$$

In this paper the optimization problem (0.5) is solved in a series of cases described below. In these cases we obtain the explicit formulas for the sharp constant  $K_{n,p}$ . In a complicated case  $n = 2m, p = \infty$  we prove a two-sided estimate for  $K_{2m,\infty}$ . In conclusion, some applications of obtained results to estimates of high order derivatives of analytic functions in subdomains of  $\mathbb{C}$  are described.

Now we describe the results of the present paper in more detail. Introduction is followed by four sections. The first of them is auxiliary. It concerns the integral

$$Q_{\mu,n,\gamma}(\beta) = \int_{-\pi/2}^{\pi/2} \left| \cos(\beta - (n+1)\varphi) \right|^\gamma \cos^\mu \varphi d\varphi, \quad (0.9)$$

depending on the parameter  $\beta$ . We consider the problem on maximum of  $Q_{\mu,n,\gamma}(\beta)$  in  $\beta$ . In what follows, by  $\mathbb{N}$  we mean the set of the natural numbers and by  $[a]$  we denote the integer part of the number  $a$ . Assuming that  $m, n \in \{0\} \cup \mathbb{N}$ ,  $m \geq n + 1$  and  $\gamma > 2 \left[ \frac{m}{n+1} \right] - 2$ , we prove the equality

$$\max_{\beta} Q_{2m,n,\gamma}(\beta) = Q_{2m,n,\gamma}(0) = \int_{-\pi/2}^{\pi/2} |\cos(n+1)\varphi|^{\gamma} \cos^{2m} \varphi d\varphi$$

and find the last integral. If  $m \leq n$  and  $\gamma > -1$ , it is shown that the function  $Q_{2m,n,\gamma}(\beta)$  is independent of  $\beta$  and its value is given.

Concretizing result of Section 1 for (0.3) with  $q = 2(m+1)/(n+1)$  and  $n \leq 2m+1$ , in Section 2 we obtain the explicit formula for  $K_{n,2(m+1)/(2m+1-n)}$ . In particular, the coefficient  $K_{n,2(m+1)/(2m+1-n)}(\alpha)$  is independent of  $\alpha$  for the case  $m \leq n$  and

$$K_{n, \frac{2(m+1)}{2m+1-n}} = \frac{n!}{\pi} \left\{ \frac{(2m-1)!!}{(2m)!!} B \left( \frac{m+1}{n+1} + \frac{1}{2}, \frac{1}{2} \right) \right\}^{\frac{n+1}{2(m+1)}}, \quad (0.10)$$

where by  $B$  is denoted the Beta-function. We note, that the above-mentioned formulas for  $K_{2m+1,\infty}$  and  $K_{n,2}$  are particular cases of (0.10) for  $n = 2m+1$  and  $n = m$ , correspondingly.

Section 3 is devoted to derivatives of even order in the case  $p = \infty$ . First, we solve the optimization problem (0.5) with  $n = 6, 8$  and  $p = \infty$ , and find the values of the sharp constants

$$K_{6,\infty} = \frac{105\sqrt{2}}{4\pi} \left( 9 \cos \frac{\pi}{28} + 3 \cos \frac{3\pi}{28} + \cos \frac{5\pi}{28} \right), \quad (0.11)$$

$$K_{8,\infty} = \frac{315}{8\pi} \left\{ 175 + 9\sqrt{2} \left( 17 \cos \frac{\pi}{36} + 9 \cos \frac{5\pi}{36} + 11 \cos \frac{7\pi}{36} \right) \right\}. \quad (0.12)$$

Further, using the result of Section 1, we obtain the two-sided estimate

$$\frac{2}{\pi} ((2m-1)!!)^2 < K_{2m,\infty} < \frac{2m}{2m-1} \frac{2}{\pi} ((2m-1)!!)^2, \quad (0.13)$$

which leads to the asymptotic formula

$$K_{2m,\infty} = \frac{2}{\pi} ((2m-1)!!)^2 + O \left( \frac{((2m-1)!!)^2}{2m-1} \right)$$

as  $m \rightarrow \infty$ .

Let us denote by

$$L_{2m} = \frac{2}{\pi} ((2m-1)!!)^2, \quad U_{2m} = \frac{2m}{2m-1} \frac{2}{\pi} ((2m-1)!!)^2$$

the values of the lower and upper bounds in two-sided estimate (0.13), correspondingly. We can compare these bounds with the sharp constant in inequality

(0.4) for  $n = 2, 4, 6, 8$  and  $p = \infty$ . Using (0.7), (0.11) and (0.12), we get

$$\begin{aligned} \frac{L_2}{K_{2,\infty}} &\approx 0.7698, & \frac{L_4}{K_{4,\infty}} &\approx 0.8830, & \frac{L_6}{K_{6,\infty}} &\approx 0.9204, & \frac{L_8}{K_{8,\infty}} &\approx 0.9396, \\ \frac{U_2}{K_{2,\infty}} &\approx 1.5396, & \frac{U_4}{K_{4,\infty}} &\approx 1.2141, & \frac{U_6}{K_{6,\infty}} &\approx 1.1045, & \frac{U_8}{K_{8,\infty}} &\approx 1.0738. \end{aligned}$$

In concluding Section 4 we collect some real-part estimates with explicit constants in the majorizing part of inequality for the modulus of derivatives of analytic functions in subdomains of  $\mathbb{C}$ .

## 1 The main lemma

First we prove the following auxiliary assertion.

**Lemma 1.** *Let  $m, n \in \{0\} \cup \mathbb{N}$ . If  $m \geq n + 1$  and  $\gamma > 2 \left\lfloor \frac{m}{n+1} \right\rfloor - 2$ , then*

$$\max_{\beta} Q_{2m,n,\gamma}(\beta) = Q_{2m,n,\gamma}(0) = \int_{-\pi/2}^{\pi/2} |\cos(n+1)\varphi|^\gamma \cos^{2m} \varphi d\varphi \quad (1.1)$$

$$= \frac{(2m-1)!!}{(2m)!!} B\left(\frac{\gamma+1}{2}, \frac{1}{2}\right) + \frac{\pi}{2^{2m+\gamma-1}(\gamma+1)} \sum_{j=1}^{\lfloor \frac{m}{n+1} \rfloor} \frac{\binom{2m}{m-j(n+1)}}{B(\frac{\gamma}{2}+j+1, \frac{\gamma}{2}-j+1)}, \quad (1.2)$$

where by  $B$  is denoted the Beta-function.

If  $m \leq n$  and  $\gamma > -1$ , then the function  $Q_{2m,n,\gamma}(\beta)$  is independent of  $\beta$ , and it is given by

$$Q_{2m,n,\gamma}(\beta) = \frac{(2m-1)!!}{(2m)!!} B\left(\frac{\gamma+1}{2}, \frac{1}{2}\right). \quad (1.3)$$

*Proof.* Making the change of variable  $\psi = \beta - (n+1)\varphi$  in (0.9) with  $\mu = 2m$ , we obtain

$$Q_{2m,n,\gamma}(\beta) = \frac{1}{n+1} \int_{\beta-(n+1)\frac{\pi}{2}}^{\beta+(n+1)\frac{\pi}{2}} |\cos \psi|^\gamma \cos^{2m} \frac{\psi - \beta}{n+1} d\psi.$$

Since the integrand is  $(n+1)\pi$ -periodic, it follows that

$$\begin{aligned} Q_{2m,n,\gamma}(\beta) &= \frac{1}{n+1} \int_0^{(n+1)\pi} |\cos \psi|^\gamma \cos^{2m} \frac{\psi - \beta}{n+1} d\psi \\ &= \frac{1}{n+1} \sum_{j=0}^n \int_{j\pi}^{(j+1)\pi} |\cos \psi|^\gamma \cos^{2m} \frac{\psi - \beta}{n+1} d\psi. \end{aligned}$$

The change of variable  $\psi - j\pi = \vartheta$  implies

$$Q_{2m,n,\gamma}(\beta) = \frac{1}{n+1} \int_0^\pi |\cos \vartheta|^\gamma g_{m,n}(\vartheta - \beta) d\vartheta, \quad (1.4)$$

where

$$g_{m,n}(\theta) = \sum_{j=0}^n \cos^{2m} \frac{\theta + j\pi}{n+1} . \quad (1.5)$$

Since

$$\cos^{2m} x = \frac{1}{2^{2m}} \left\{ \binom{2m}{m} + 2 \sum_{k=0}^{m-1} \binom{2m}{k} \cos 2(m-k)x \right\} ,$$

we can write (1.5) in the form

$$g_{m,n}(\theta) = \frac{n+1}{2^{2m}} \binom{2m}{m} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \sum_{j=0}^n \cos \frac{2(m-k)(\theta + j\pi)}{n+1} .$$

Putting here  $k = m - l$  ( $l = 1, 2, \dots, m$ ) and taking into account that

$$\frac{1}{2^{2m}} \binom{2m}{m} = \frac{(2m)!}{2^{2m}(m!)^2} = \frac{(2m)!}{((2m)!!)^2} = \frac{(2m-1)!!}{(2m)!!} ,$$

we obtain

$$g_{m,n}(\theta) = \frac{(2m-1)!!(n+1)}{(2m)!!} + \frac{1}{2^{2m-1}} \sum_{l=1}^m \binom{2m}{m-l} \sum_{j=0}^n \cos \frac{2l(\theta + j\pi)}{n+1} . \quad (1.6)$$

Consider the interior sum in (1.6). We have

$$\sum_{j=0}^n \cos \frac{2l(\theta + j\pi)}{n+1} = \Re \left\{ e^{\frac{2l\theta i}{n+1}} \sum_{j=0}^n e^{\frac{2l\pi i j}{n+1}} \right\} .$$

So, for  $l \in \{1, \dots, m\}$ ,

$$\sum_{j=0}^n \cos \frac{2l(\theta + j\pi)}{n+1} = \Re \left\{ e^{\frac{2l\theta i}{n+1}} \frac{1 - e^{\frac{2l\pi i (n+1)}{n+1}}}{1 - e^{\frac{2l\pi i}{n+1}}} \right\} = 0 , \quad (1.7)$$

if  $s = \frac{l}{n+1} \notin \mathbb{N}$ , and

$$\sum_{j=0}^n \cos \frac{2l(\theta + j\pi)}{n+1} = (n+1) \cos 2s\theta , \quad (1.8)$$

if  $s = \frac{l}{n+1} \in \mathbb{N}$ .

Taking into account (1.7) and (1.8), we can rewrite (1.6) in the form

$$g_{m,n}(\theta) = \frac{(2m-1)!!(n+1)}{(2m)!!} + \frac{n+1}{2^{2m-1}} \sum_{s=1}^{\lfloor \frac{m}{n+1} \rfloor} \binom{2m}{m-s(n+1)} \cos 2s\theta .$$

Combining this with (1.4), we obtain

$$Q_{2m,n,\gamma}(\beta) = \frac{(2m-1)!!}{(2m)!!} \int_0^\pi |\cos \vartheta|^\gamma d\vartheta \\ + \frac{1}{2^{2m-1}} \sum_{s=1}^{\lfloor \frac{m}{n+1} \rfloor} \binom{2m}{m-s(n+1)} \int_0^\pi |\cos \vartheta|^\gamma \cos 2s(\vartheta - \beta) d\vartheta .$$

Since the integrands in the last equality are  $\pi$ -periodic, it follows that

$$Q_{2m,n,\gamma}(\beta) = \frac{(2m-1)!!}{(2m)!!} \int_{-\pi/2}^{\pi/2} \cos^\gamma \vartheta d\vartheta \\ + \frac{1}{2^{2m-1}} \sum_{s=1}^{\lfloor \frac{m}{n+1} \rfloor} \binom{2m}{m-s(n+1)} \int_{-\pi/2}^{\pi/2} \cos^\gamma \vartheta \cos 2s(\vartheta - \beta) d\vartheta ,$$

that is

$$Q_{2m,n,\gamma}(\beta) = \frac{2(2m-1)!!}{(2m)!!} \int_0^{\pi/2} \cos^\gamma \vartheta d\vartheta \\ + \frac{1}{2^{2(m-1)}} \sum_{s=1}^{\lfloor \frac{m}{n+1} \rfloor} \binom{2m}{m-s(n+1)} \cos 2s\beta \int_0^{\pi/2} \cos^\gamma \vartheta \cos 2s\vartheta d\vartheta . \quad (1.9)$$

Let  $m \geq n+1$ . Taking into account the formula (see, e.g., Gradshteyn and Ryzhik [1], **3.631(9)**)

$$\int_0^{\pi/2} \cos^{\nu-1} x \cos ax dx = \frac{\pi}{2^\nu \nu B\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)} , \quad (1.10)$$

where  $\Re \nu > 0$ , and the condition  $\gamma > 2 \lfloor \frac{m}{n+1} \rfloor - 2$  of the present lemma, we conclude that

$$\int_0^{\pi/2} \cos^\gamma \vartheta \cos 2s\vartheta d\vartheta > 0$$

for any  $s \in \left\{1, 2, \dots, \lfloor \frac{m}{n+1} \rfloor\right\}$ . This and (1.9) imply that the maximum of  $Q_{2m,n,\gamma}(\beta)$  in  $\beta$  is attained at  $\beta = 0$ . Hence, by (0.9) we obtain (1.1). Calculating by (1.10) the integrals in (1.9), we arrive at (1.2).

The sum in (1.9) vanishes in the case  $m \leq n$ . Hence, the function  $Q_{2m,n,\gamma}(\beta)$  is independent of  $\beta$  under condition  $m \leq n$ , which proves (1.3).  $\square$

## 2 Sharp estimates for derivatives of analytic functions with $\Re f \in h^p(\mathbb{R}_+^2)$ , $p = 2(m+1)/(2m+1-n)$

In what follows, by  $h^p(\mathbb{R}_+^2)$ ,  $1 \leq p \leq \infty$ , we mean the Hardy space of harmonic functions in the upper half-plane  $\mathbb{R}_+^2$  which are represented by the Poisson inte-

gral with a density in  $L^p(-\infty, \infty)$ . It is well known (see, e.g. Levin [11], Sect. 19.3) that  $f$  belongs to the Hardy space  $H^p(\mathbb{C}_+)$  of analytic functions in  $\mathbb{C}_+$  if  $\Re f \in h^p(\mathbb{R}_+^2)$ ,  $1 < p < \infty$ . Besides, any function  $f \in H^p(\mathbb{C}_+)$ ,  $1 < p < \infty$ , admits the representation (0.2) since  $\Re f \in h^p(\mathbb{R}_+^2)$ .

Now we consider the case  $p = 2(m+1)/(2m+1-n)$  in inequality (0.4), that is  $q = 2(m+1)/(n+1)$ . We suppose that  $n \leq 2m+1$ ,  $n \geq 1$ . In the case  $n = 2m+1$  we put  $p = \infty$ .

**Theorem 1.** *Let  $\Re f \in h^p(\mathbb{R}_+^2)$  with  $p = 2(m+1)/(2m+1-n)$ , and let  $z$  be an arbitrary point in  $\mathbb{C}_+$ . The sharp constant  $K_{n, 2(m+1)/(2m+1-n)}$  in the inequality*

$$|f^{(n)}(z)| \leq \frac{K_{n, 2(m+1)/(2m+1-n)}}{(\Im z)^{n+\frac{1}{p}}} \|\Re f\|_p \quad (2.1)$$

is given by

$$\begin{aligned} K_{n, \frac{2(m+1)}{2m+1-n}} &= \frac{n!}{\pi} \left\{ \int_{-\pi/2}^{\pi/2} |\cos(n+1)\varphi|^{\frac{2(m+1)}{n+1}} \cos^{2m} \varphi d\varphi \right\}^{\frac{n+1}{2(m+1)}} \\ &= \frac{n!}{\pi} \left\{ \frac{(2m-1)!!}{(2m)!!} B\left(\frac{m+1}{n+1} + \frac{1}{2}, \frac{1}{2}\right) \right. \\ &\quad \left. + \frac{\pi(n+1)}{2^{2m-1+\frac{2(m+1)}{n+1}}(2m+n+3)} \sum_{j=1}^{\lfloor \frac{m}{n+1} \rfloor} \frac{\binom{2m}{m-j(n+1)}}{B\left(\frac{m+1}{n+1} + j+1, \frac{m+1}{n+1} - j+1\right)} \right\}^{\frac{n+1}{2(m+1)}}. \end{aligned} \quad (2.2)$$

In particular,

$$K_{n, \frac{2(m+1)}{2m+1-n}} = \frac{n!}{\pi} \left\{ \frac{(2m-1)!!}{(2m)!!} B\left(\frac{m+1}{n+1} + \frac{1}{2}, \frac{1}{2}\right) \right\}^{\frac{n+1}{2(m+1)}} \quad (2.4)$$

for  $m \leq n$ .

*Proof.* Putting  $q = 2(m+1)/(n+1)$  in (0.3)-(0.5) and  $\gamma = 2(m+1)/(n+1)$ ,  $\mu = 2m$  in (0.9), we can write the sharp constant  $K_{n,p}$  in inequality (2.1) as follows

$$\begin{aligned} K_{n, \frac{2(m+1)}{2m+1-n}} &= \frac{n!}{\pi} \max_{\alpha} \left\{ Q_{2m, n, \frac{2(m+1)}{n+1}} \left( \alpha + \frac{n\pi}{2} \right) \right\}^{\frac{n+1}{2(m+1)}} \\ &= \frac{n!}{\pi} \max_{\beta} \left\{ Q_{2m, n, \frac{2(m+1)}{n+1}} (\beta) \right\}^{\frac{n+1}{2(m+1)}}. \end{aligned} \quad (2.5)$$

For  $\gamma = 2(m+1)/(n+1)$  we have

$$\frac{\gamma}{2} + 1 = \frac{m+1}{n+1} + 1 > \left\lfloor \frac{m}{n+1} \right\rfloor + 1$$

that is, the condition

$$\gamma > 2 \left\lfloor \frac{m}{n+1} \right\rfloor - 2$$

of Lemma 1 is satisfied. Applying Lemma 1 to (2.5), we complete the proof.  $\square$

Two following consequences of Theorem 1 contain explicit formulas for  $K_{n,p}$  in particular cases.

**Corollary 1.** *If  $n = 2m$ ,  $m \in \mathbb{N}$ , then*

$$K_{2m,2m+2} = \frac{(2m)!}{\pi} \left\{ \frac{\sqrt{\pi}(2m-1)!! \Gamma\left(\frac{m+1}{2m+1} + \frac{1}{2}\right)}{(2m)!! \Gamma\left(\frac{m+1}{2m+1} + 1\right)} \right\}^{\frac{2m+1}{2(m+1)}}. \quad (2.6)$$

**Corollary 2.** *If  $m = k(n+1) - 1$ ,  $k \in \mathbb{N}$ , then*

$$\begin{aligned} K_{n, \frac{2k}{2k-1}} &= \frac{n!}{\pi} \left\{ \frac{\sqrt{\pi}(2k(n+1)-3)!! \Gamma(k + \frac{1}{2})}{(2k(n+1)-2)!! k!} \right. \\ &\quad \left. + \frac{\pi}{2^{2k(n+2)-3}} \sum_{j=1}^{k-1} \binom{2k(n+1)-2}{(k-j)(n+1)-1} \binom{2k}{k-j} \right\}^{\frac{1}{2k}}. \end{aligned}$$

### 3 Estimates for even order derivatives of analytic functions with $\Re f \in h^\infty(\mathbb{R}_+^2)$

By (0.3),

$$K_{2m,\infty}(\alpha) = \frac{(2m)!}{\pi} \int_{-\pi/2}^{\pi/2} |\cos(\alpha - (2m+1)\varphi)| \cos^{2m-1} \varphi d\varphi. \quad (3.1)$$

The starting point of this section is the following assertion from the paper by Kresin and Maz'ya [8].

**Lemma 2.** *The equality*

$$\frac{dK_{2m,\infty}}{d\alpha} = \frac{(2m)!}{\pi(2m+1)^2 2^{2(m-1)}} \int_0^{\pi/2} (|\cos(\alpha - \varphi)| - |\cos(\alpha + \varphi)|) \Lambda_m(\varphi) d\varphi \quad (3.2)$$

holds with

$$\Lambda_m(\varphi) = \sum_{\ell=1}^m (-1)^\ell (2\ell-1) \binom{2m-1}{m-\ell} \frac{\sin \frac{(2\ell-1)\varphi}{2m+1}}{\sin \frac{(2\ell-1)\pi}{2(2m+1)}}. \quad (3.3)$$



**Remark 1.** Before passing to applications of Lemma 2 we make two remarks. The first one concerns the range of  $\beta$  in the evaluation of the maximum

$$\max_{\beta} Q_{\mu,n,\gamma}(\beta),$$

where the function  $Q_{\mu,n,\gamma}(\beta)$  is defined by (0.9). It is clear, that  $Q_{\mu,n,\gamma}(\beta)$  is  $\pi$ -periodic and even function in  $\beta$ . Therefore, we can limit our consideration of  $Q_{\mu,n,\gamma}(\beta)$  to the interval  $[0, \pi/2]$ .

The second remark relates the sign of the function  $|\cos(\beta - \varphi)| - |\cos(\beta + \varphi)|$ , which appear inside of integral (3.2). We show that

$$|\cos(\beta - \varphi)| \geq |\cos(\beta + \varphi)| \quad (3.4)$$

for  $\beta, \varphi \in [0, \pi/2]$ . In fact, since

$$|\cos(\beta - \varphi)| - |\cos(\beta + \varphi)| = \begin{cases} \cos(\beta - \varphi) - \cos(\beta + \varphi) & \text{for } \varphi \in [0, \frac{\pi}{2} - \beta], \\ \cos(\beta - \varphi) + \cos(\beta + \varphi) & \text{for } \varphi \in (\frac{\pi}{2} - \beta, \frac{\pi}{2}], \end{cases}$$

it follows that

$$|\cos(\beta - \varphi)| - |\cos(\beta + \varphi)| = \begin{cases} 2 \sin \varphi \sin \beta & \text{for } \varphi \in [0, \frac{\pi}{2} - \beta], \\ 2 \cos \varphi \cos \beta & \text{for } \varphi \in (\frac{\pi}{2} - \beta, \frac{\pi}{2}], \end{cases}$$

and hence (3.4) holds for  $\beta, \varphi \in [0, \pi/2]$ . Besides, the equality sign in (3.4) holds only for  $\beta = 0$  or for  $\beta = \pi/2$  provided that  $\varphi \in (0, \pi/2)$ .

In the next two assertions we deal with the values of constants  $K_{6,\infty}$  and  $K_{8,\infty}$ .

**Corollary 3.** Let  $\Re f \in h^\infty(\mathbb{R}_+^2)$ , and let  $z$  be an arbitrary point in  $\mathbb{C}_+$ . The sharp constant  $K_{6,\infty}$  in the inequality

$$|f^{(6)}(z)| \leq \frac{K_{6,\infty}}{(\Im z)^6} \|\Re f\|_\infty \quad (3.5)$$

is given by

$$K_{6,\infty} = \frac{105\sqrt{2}}{4\pi} \left( 9 \cos \frac{\pi}{28} + 3 \cos \frac{3\pi}{28} + \cos \frac{5\pi}{28} \right). \quad (3.6)$$

*Proof.* By Lemma 2,

$$\frac{dK_{6,\infty}}{d\alpha} = \frac{45}{49\pi} \int_0^{\pi/2} (|\cos(\alpha - \varphi)| - |\cos(\alpha + \varphi)|) \Lambda_3(\varphi) d\varphi, \quad (3.7)$$

where

$$\Lambda_3(\varphi) = 5 \left( -2 \frac{\sin \frac{\varphi}{7}}{\sin \frac{\pi}{14}} + 3 \frac{\sin \frac{3\varphi}{7}}{\sin \frac{3\pi}{14}} - \frac{\sin \frac{5\varphi}{7}}{\sin \frac{5\pi}{14}} \right). \quad (3.8)$$

Using the identities  $\sin 3x = 3 \sin x - 4 \sin^3 x$ ,  $\sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x$  in (3.8), we find

$$\Lambda_3(\varphi) = 80 \frac{\sin \frac{\pi}{14} \sin \frac{\varphi}{7}}{\sin \frac{3\pi}{14} \sin \frac{5\pi}{14}} \left( \sin^2 \frac{\pi}{14} - \sin^2 \frac{\varphi}{7} \right) F_3(\varphi), \quad (3.9)$$

where

$$F_3(\varphi) = \left( 8 \sin^2 \frac{\pi}{14} - 7 \right) \sin^2 \frac{\pi}{14} + \left( 3 - 4 \sin^2 \frac{\pi}{14} \right) \sin^2 \frac{\varphi}{7}.$$

Since  $3 - 4 \sin^2(\pi/14) > 3 - 4 \sin^2(\pi/6) > 0$ , we have

$$F_3(\varphi) < \left( 8 \sin^2 \frac{\pi}{14} - 7 \right) \sin^2 \frac{\pi}{14} + \left( 3 - 4 \sin^2 \frac{\pi}{14} \right) \sin^2 \frac{\pi}{14} = -4 \cos^2 \frac{\pi}{14} \sin^2 \frac{\pi}{14},$$

which together with (3.9) proves the inequality  $\Lambda_3(\varphi) < 0$  for  $\varphi \in (0, \pi/2)$ . Now, by (3.7) and (3.4) we conclude that

$$\frac{dK_{6,\infty}}{d\alpha} < 0$$

for  $\alpha \in (0, \pi/2)$ . Thus, by (3.1),

$$\begin{aligned} K_{6,\infty} &= K_{6,\infty}(0) = \frac{6!}{\pi} \int_{-\pi/2}^{\pi/2} |\cos 7\varphi| \cos^5 \varphi \, d\varphi = 2 \frac{6!}{\pi} \left\{ \int_0^{\pi/14} \cos 7\varphi \cos^5 \varphi \, d\varphi \right. \\ &\quad \left. - \int_{\pi/14}^{3\pi/14} \cos 7\varphi \cos^5 \varphi \, d\varphi + \int_{3\pi/14}^{5\pi/14} \cos 7\varphi \cos^5 \varphi \, d\varphi - \int_{5\pi/14}^{\pi/2} \cos 7\varphi \cos^5 \varphi \, d\varphi \right\}. \end{aligned}$$

Evaluating the integrals on the right-hand side of the last equality, we arrive at (3.6).  $\square$

**Corollary 4.** *Let  $\Re f \in h^\infty(\mathbb{R}_+^2)$ , and let  $z$  be an arbitrary point in  $\mathbb{C}_+$ . The sharp constant  $K_{8,\infty}$  in the inequality*

$$|f^{(8)}(z)| \leq \frac{K_{8,\infty}}{(\Im z)^8} \|\Re f\|_\infty \quad (3.10)$$

is given by

$$K_{8,\infty} = \frac{315}{8\pi} \left\{ 175 + 9\sqrt{2} \left( 17 \cos \frac{\pi}{36} + 9 \cos \frac{5\pi}{36} + 11 \cos \frac{7\pi}{36} \right) \right\}. \quad (3.11)$$

*Proof.* By Lemma 2,

$$\frac{dK_{8,\infty}}{d\alpha} = \frac{70}{9\pi} \int_0^{\pi/2} (|\cos(\alpha - \varphi)| - |\cos(\alpha + \varphi)|) \Lambda_4(\varphi) \, d\varphi, \quad (3.12)$$

where

$$\Lambda_4(\varphi) = 7 \left( -5 \frac{\sin \frac{\varphi}{9}}{\sin \frac{\pi}{18}} + 18 \sin \frac{3\varphi}{9} - 5 \frac{\sin \frac{5\varphi}{9}}{\sin \frac{5\pi}{18}} + \frac{\sin \frac{7\varphi}{9}}{\sin \frac{7\pi}{18}} \right). \quad (3.13)$$

Using the identities  $\sin 3x = 3 \sin x - 4 \sin^3 x$ ,  $\sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x$  and  $\sin 7x = 7 \sin x - 56 \sin^3 x + 112 \sin^5 x - 64 \sin^7 x$  in (3.8), we find

$$\Lambda_4(\varphi) = 896 \frac{\sin^2 \frac{\pi}{18} \sin \frac{\varphi}{9}}{\sin \frac{5\pi}{18} \sin \frac{7\pi}{18}} \left( \sin^2 \frac{\pi}{18} - \sin^2 \frac{\varphi}{9} \right) F_4(\varphi), \quad (3.14)$$

where

$$\begin{aligned} F_4(\varphi) = & \left( 155 - 604 \sin^2 \frac{\pi}{18} + 768 \sin^4 \frac{\pi}{18} - 320 \sin^6 \frac{\pi}{18} \right) \sin^4 \frac{\pi}{18} \\ & + \left( -90 + 396 \sin^2 \frac{\pi}{18} - 560 \sin^4 \frac{\pi}{18} + 256 \sin^6 \frac{\pi}{18} \right) \sin^2 \frac{\pi}{18} \sin^2 \frac{\varphi}{9} \\ & + \left( 15 - 80 \sin^2 \frac{\pi}{18} + 128 \sin^4 \frac{\pi}{18} - 64 \sin^6 \frac{\pi}{18} \right) \sin^4 \frac{\varphi}{9}. \end{aligned}$$

It follows

$$\begin{aligned} F_4(\varphi) & > \left( 155 - 604 \sin^2 \frac{\pi}{18} + 768 \sin^4 \frac{\pi}{18} - 320 \sin^6 \frac{\pi}{18} \right) \sin^4 \frac{\pi}{18} \\ & - 10 \left( 9 + 56 \sin^4 \frac{\pi}{18} \right) \sin^4 \frac{\pi}{18} - 16 \left( 5 + 4 \sin^4 \frac{\pi}{18} \right) \sin^6 \frac{\pi}{18} \\ & = \left( 65 - 684 \sin^2 \frac{\pi}{18} + 208 \sin^4 \frac{\pi}{18} - 384 \sin^6 \frac{\pi}{18} \right) \sin^4 \frac{\pi}{18}. \end{aligned}$$

Using this inequality and  $\sin(\pi/12) = (\sqrt{6} - \sqrt{2})/4$ , we obtain

$$\begin{aligned} F_4(\varphi) & > \left( 65 - 684 \sin^2 \frac{\pi}{18} - 384 \sin^6 \frac{\pi}{18} \right) \sin^4 \frac{\pi}{18} \\ & > \left( 65 - 684 \sin^2 \frac{\pi}{12} - 384 \sin^6 \frac{\pi}{12} \right) \sin^4 \frac{\pi}{18} = (261\sqrt{3} - 433) \sin^4 \frac{\pi}{18} > 0. \end{aligned}$$

The last estimate together with (3.14) proves the inequality  $\Lambda_4(\varphi) > 0$  for  $\varphi \in (0, \pi/2)$ . Now, by (3.12) and (3.4) we conclude that

$$\frac{dK_{8,\infty}}{d\alpha} > 0$$

for  $\alpha \in (0, \pi/2)$ . Thus, by (3.1),

$$\begin{aligned} K_{8,\infty} = K_{8,\infty}(\pi/2) &= \frac{8!}{\pi} \int_{-\pi/2}^{\pi/2} |\sin 9\varphi| \cos^7 \varphi \, d\varphi = 2 \frac{8!}{\pi} \left\{ \int_0^{\pi/9} \sin 9\varphi \cos^7 \varphi \, d\varphi \right. \\ & - \int_{\pi/9}^{2\pi/9} \sin 9\varphi \cos^7 \varphi \, d\varphi + \int_{2\pi/9}^{\pi/3} \sin 9\varphi \cos^7 \varphi \, d\varphi \\ & \left. - \int_{\pi/3}^{4\pi/9} \sin 9\varphi \cos^7 \varphi \, d\varphi + \int_{4\pi/9}^{\pi/2} \sin 9\varphi \cos^7 \varphi \, d\varphi \right\}. \end{aligned}$$

After evaluating the integrals on the right-hand side of the last equality, we arrive at (3.11).  $\square$

Now we apply Lemma 1 in the proof of the following assertion.

**Theorem 2.** *The following two-side inequality*

$$\frac{2}{\pi}((2m-1)!!)^2 < K_{2m,\infty} < \frac{2m}{2m-1} \frac{2}{\pi}((2m-1)!!)^2 \quad (3.15)$$

holds.

*Proof.* By (3.1),

$$K_{2m,\infty}(\alpha) < \frac{(2m)!}{\pi} \int_{-\pi/2}^{\pi/2} |\cos(\alpha - (2m+1)\varphi)| \cos^{2(m-1)} \varphi \, d\varphi.$$

From this and equality (1.3) with  $m-1$  instead of  $m$  and  $n = 2m, \gamma = 1$ , we obtain

$$K_{2m,\infty}(\alpha) < \frac{2(2m)!}{\pi} \frac{(2m-3)!!}{(2m-2)!!} = \frac{2}{\pi} \frac{2m}{2m-1} \frac{(2m-1)!(2m-1)!!}{(2m-2)!!}, \quad (3.16)$$

which together with (0.5) proves the upper estimate in (3.15).

Now we turn to the inverse estimate of  $K_{2m,\infty}$  in (3.15). It follows from (0.5) and (3.1),

$$K_{2m,\infty} \geq K_{2m,\infty}(\alpha) > \frac{(2m)!}{\pi} \int_{-\pi/2}^{\pi/2} |\cos(\alpha - (2m+1)\varphi)| \cos^{2m} \varphi \, d\varphi.$$

Using the last estimate and (1.3) with  $n = 2m$  and  $\gamma = 1$ , we find that

$$K_{2m,\infty} > 2 \frac{(2m)!}{\pi} \frac{(2m-1)!!}{(2m)!!},$$

which is equivalent to the lower estimate in (3.15).  $\square$

In final section we describe some real-part estimates which take the explicit form by combination with formulas for  $K_{n,p}$  from (0.6)-(0.8), Theorem 1, Corollaries 1-4 and the estimate of Theorem 2.

## 4 Explicit estimates for derivatives of analytic functions in domains

The next two assertions were proved in paper by Kresin and Maz'ya [10] .

**Proposition 1.** *Let  $\Omega = \mathbb{C} \setminus \overline{G}$ , where  $G$  is a convex domain in  $\mathbb{C}$ , and let  $f$  be a holomorphic function in  $\Omega$  with bounded real part. Then for any point  $z \in \Omega$  the inequality*

$$|f^{(n)}(z)| \leq \frac{K_{n,\infty}}{d_z^n} \sup_{\Omega} |\Re f|, \quad n = 1, 2, \dots,$$

holds with  $d_z = \text{dist}(z, \partial\Omega)$ , where

$$K_{n,\infty} = \frac{n!}{\pi} \max_{\beta} \int_{-\pi/2}^{\pi/2} |\cos(\beta - (n+1)\varphi)| \cos^{n-1} \varphi \, d\varphi$$

is the best constant in the inequality

$$|f^{(n)}(z)| \leq \frac{K_{n,\infty}}{(\Im z)^n} \|\Re f\|_{\infty}$$

for holomorphic functions  $f$  in the half-plane  $\mathbb{C}_+$  with the bounded real part.

**Proposition 2.** *Let  $\Omega$  be a domain in  $\mathbb{C}$ , and let  $\Re(\Omega)$  be the set of holomorphic functions  $f$  in  $\Omega$  with  $\sup_{\Omega} |\Re f| \leq 1$ . Assume that a point  $\zeta \in \partial\Omega$  can be touched by an interior disk  $D$ . Then*

$$\limsup_{z \rightarrow \zeta} \sup_{f \in \Re(\Omega)} |z - \zeta|^n |f^{(n)}(z)| \leq K_{n,\infty}, \quad n = 1, 2, \dots,$$

where  $z$  is a point of the radius of  $D$  directed from the center to  $\zeta$ . Here the constant  $K_{n,\infty}$  is the same as in Proposition 1 and cannot be diminished.

By (0.6) and (3.15), the constant  $K_{n,\infty}$  in Propositions 1 and 2 obeys the relations

$$K_{2m-1,\infty} = \frac{2}{\pi} \frac{((2m-1)!!)^2}{2m-1}, \quad K_{2m,\infty} < \frac{2m}{2m-1} \frac{2}{\pi} ((2m-1)!!)^2$$

for any  $m \in \mathbb{N}$ . The values of the constants  $K_{2,\infty}$ ,  $K_{4,\infty}$ ,  $K_{6,\infty}$  and  $K_{8,\infty}$  in these statements are given by (0.7), (3.6) and (3.11), correspondingly.

Now we turn to a real-part estimate for derivatives of analytic functions in the disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . By  $h^p(\mathbb{D})$ ,  $1 \leq p \leq \infty$ , we mean the Hardy space of harmonic functions in the real unit disk  $\mathbb{D}$  which are represented by the Poisson integral with a density in  $L^p(\partial\mathbb{D})$ . Below by  $\|\cdot\|_p$  we denote the norm in the space  $L^p(\partial\mathbb{D})$ .

The inequality, obtained by Khavinson [5]

$$|f'(z)| \leq \frac{4}{\pi(1-r^2)} \|\Re f\|_{\infty},$$

contains the best possible coefficient in front of  $\|\Re f\|_{\infty}$ , where  $r = |z| < 1$ .

The next estimate for derivatives of analytic functions with  $\Re f \in h^p(\mathbb{D})$

$$|f^{(n)}(z)| \leq \frac{C_{n,p}}{(1-r^2)^{n+\frac{1}{p}}} \|\Re f\|_p \quad (4.1)$$

was proved in the paper by Kalaj and Elkie [3]. The representation of the constant  $2^{-(n+\frac{1}{p})} C_{n,p}$  in [3] is equivalent to the representation (0.5) with (0.3) for the sharp constant  $K_{n,p}$  in inequality (0.4). The case  $n = 1$  in (4.1) was considered by Kalaj and Marković [4].

The assertion below was established in paper by Kresin [9].

**Proposition 3.** *Let  $f$  be an analytic function in  $\mathbb{D}$  with  $\Re f \in h^p(\mathbb{D})$ . The inequality*

$$\sup_{|z|<1} \sup_{\|\Re f\|_p \leq 1} (1 - |z|^2)^{n+\frac{1}{p}} |f^{(n)}(z)| \geq 2^{n+\frac{1}{p}} K_{n,p}$$

*holds, where  $K_{n,p}$  is the sharp constant in inequality (0.4).*

Proposition 3 together with (4.1) leads to relation

$$\sup_{|z|<1} \sup_{\|\Re f\|_p \leq 1} (1 - |z|^2)^{n+\frac{1}{p}} |f^{(n)}(z)| = 2^{n+\frac{1}{p}} K_{n,p},$$

which shows that the constant  $C_{n,p} = 2^{n+\frac{1}{p}} K_{n,p}$  in estimate (4.1) cannot be diminished.

The explicit expression for  $C_{2m-1,\infty}$  was established by Kalaj and Elkie [3]. The formulas for  $C_{2m,\infty} = 2^{2m} K_{2m,\infty}$  with  $m = 1, 2, 3, 4$  can be obtained by (0.7), (3.6) and (3.11). Other examples of the explicit formulas for the constant  $C_{n,p}$  in (4.1) can be derived by relation  $C_{n,p} = 2^{n+\frac{1}{p}} K_{n,p}$  and Theorem 1 as well as Corollaries 1, 2.

The next two-sided inequality

$$\frac{2^{2m+1}}{\pi} ((2m-1)!!)^2 < C_{2m,\infty} < \frac{2m}{2m-1} \frac{2^{2m+1}}{\pi} ((2m-1)!!)^2$$

follows from equality  $C_{2m,\infty} = 2^{2m} K_{2m,\infty}$  and estimate (3.15). This implies

$$C_{2m,\infty} \sim \frac{2^{2m+1}}{\pi} ((2m-1)!!)^2$$

as  $m \rightarrow \infty$ .

## References

- [1] I.S. Gradshteyn and I.M. Ryzhik; A. Jeffrey, editor, *Table of Integrals, Series and Products*, Fifth edition, Academic Press, New York, 1994.
- [2] J. Hadamard, *Sur les fonctions entières de la forme  $e^{G(X)}$* , C.R. Acad. Sci., **114** (1892), 1053–1055.
- [3] D. Kalaj and N. D. Elkie, *On real part theorem for the higher derivatives of analytic functions in the unit disk*, Comp. Meth. and Function Theory, **13**:2 (2013), 189–203.
- [4] D. Kalaj and M. Marković, *Optimal estimates for the gradient of harmonic functions in the unit disk*, Complex Analysis and Operator Theory, **7**:4 (2013), 1167–1183.
- [5] D. Khavinson, *An extremal problem for harmonic functions in the ball*, Canad. Math. Bull., **35**:2 (1992), 218–220.

- [6] G. Kresin and V. Maz'ya, *Sharp Real-Part Theorems. A Unified Approach*. Lect. Notes in Math., **1903**, Springer-Verlag, Berlin-Heidelberg-New York, 2007.
- [7] G. Kresin and V. Maz'ya, *Sharp real-part theorems in the upper half-plane and similar estimates for harmonic functions*, J. Math. Sc. (New York), **179**:1 (2011), 141–163.
- [8] G. Kresin and V. Maz'ya, *Sharp real-part theorems for high order derivatives*, J. Math. Sc. (New York), **181**:2 (2012), 107–125.
- [9] G. Kresin, *Sharp and maximized real-part estimates for derivatives of analytic functions in the disk*, Rendiconti Lincei - Matematica e Applicazioni, **24**:1 (2013), 95–110.
- [10] G. Kresin and V. Maz'ya, *Optimal estimates for derivatives of solutions to Laplace, Lamé and Stokes equations*, J. Math. Sc. (New York), **196**:3 (2014), 300–321.
- [11] B.Ya. Levin, *Lectures on Entire Functions*, Transl. of Math. Monographs, v. 150, Amer. Math. Soc., Providence, 1996.